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A family of discrete-time exactly-solvable exclusion processes on a one-dimensional lattice

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Abstract. A two-parameter family of discrete-time exactly-solvable exclusion processes on a one-dimensional lattice is introduced, which contains the asymmetric simple exclusion process and the drop-push model as particular cases. The process is rewritten in terms of boundary conditions, and the conditional probabilities are calculated using the Bethe-ansatz. This is the discrete-time version of the continuous-time processes already investigated in [1–3]. The drift- and diffusion-rates of the particles are also calculated for the two-particle sector.

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1 Introduction

The asymmetric exclusion process and the problems related to it, including for example bipolymerization [4], dynamical models of interface growth [5], traffic models [6], the noisy Burgers equation [7], and the study of shocks [8,9], have been extensively studied in recent years. The dynamical properties of this model have been studied in [9,10]. As the results obtained by approaches like mean field are not reliable in one dimension, it is useful to introduce solvable models and analytic methods to extract exact physical results.

The totally asymmetric simple exclusion model on a one-dimensional lattice is one of the simplest examples from which exact results can be obtained. Such systems consist of a lattice in which every site is either empty or occupied by a single particle. Particles can hop to the right, if their right-neighboring site is empty. The steady-state of such systems have been extensively studied, for continuous-time as well as discrete-time evolutions. Among the methods used to study the steady-state properties of such systems is the matrix-product ansatz, [11–15]. Various methods have also been used to study the time-dependent state of such systems. In [16–18], generalizations of the matrix-product ansatz have been used to study asymmetric exclusion processes. In [19], an explicit form for the conditional probability of

finding particles on specific sites in a system of asymmetric exclusion process was obtained in terms of a determinant.

In [19], the coordinate Bethe-ansatz is used to solve the asymmetric simple exclusion process on a one-dimensional lattice. In [1], a similar technique was used to solve the drop-push model, and a generalized one-parameter model interpolating between the asymmetric simple exclusion model and the drop-push model. In [2], this family was further generalized to a family of processes with arbitrary left- and right-diffusion rates. All of these models were lattice models. The behaviour of the latter model on a continuum was investigated in [3]. The discrete-time version of the asymmetric exclusion process was discussed in [23]. In [20–22], a similar Bethe-ansatz approach was used to study exclusion systems consisting of several kinds of particles.

Here we consider discrete-time asymmetric exclusion processes in a one-dimensional lattice. The scheme of the paper is the following. In Section 2, a system is introduced which consists of a one-dimensional lattice in which each of the sites are either empty or occupied by a single particle. A discrete-time evolution is introduced and it is shown that the interaction between particles can be substituted by a suitable boundary condition. In Section 3, the conditional probability of occupied sites is obtained. In Section 4, the drift rates for the two particle sector are calculated. In Section 5, the diffusion rate for the two particle sector is calculated. Section 6 is devoted to the concluding remarks.

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It is seen that for large times, the results of the continuous-time evolution are recovered, namely that the drift rates tend to the no-interaction drift rates, while the diffusion rate is generally larger than the diffusion rate of the non-interacting system.

2 A family of discrete-time exclusion processes on a one-dimensional lattice

Consider a one-dimensional lattice, in which each site is either empty or occupied by one particle. The probability that the first particle is in x_1, the second particle is in x_2, etc. is denoted by

$$P(x_1, x_2, \dots), \qquad x_1 < x_2 < \cdots$$

The process is that each particle can hop to the right, with the probability α, if the its right-hand side neighbor is empty:

$$A\emptyset \to \emptyset A, \qquad \text{with the probability } \alpha. \qquad (1)$$

Consider the following evolution equation and boundary condition for the two-particle sector.

$$P(x_1, x_2, t+1) = (1-\alpha)^2 P(x_1, x_2, t)$$
$$+ \alpha(1-\alpha)\left[P(x_1 - 1, x_2, t) + P(x_1, x_2 - 1, t)\right]$$
$$+ \alpha^2 P(x_1 - 1, x_2 - 1, t), \qquad x_1 < x_2, \quad (2)$$

and

$$P(x, x) = \lambda P(x, x+1) + \mu P(x-1, x), \qquad \lambda + \mu = 1. \quad (3)$$

Equation (2) describes a system with a diffusion process which occurs simultaneously for all particles. This is in contrast to a system for which at each step only one particle can hop to the right (if its right-hand site is empty). In the latter case, terms proportional to α^2 would be omitted from the above equation.

Using (3), it is seen that

$$P(x, x+1, t+1) = \left[(1-\alpha)^2 + \lambda\alpha(1-\alpha)\right] P(x, x+1, t)$$
$$+ \alpha(1-\alpha) P(x-1, x+1, t)$$
$$+ \left[\alpha^2 + \mu\alpha(1-\alpha)\right] P(x-1, x, t). \quad (4)$$

So it is seen that (2) and (3) describe a system where particles can push:

$$AA\emptyset \to \emptyset AA, \qquad \text{with the probability } \beta \qquad (5)$$

where

$$\beta = \mu\alpha(1-\alpha) + \alpha^2 = \alpha - \lambda\alpha(1-\alpha). \qquad (6)$$

One can use (2) and (3), to obtain pushing rates in multi-particle sectors as well. This is especially simple in

two cases: $(\lambda = 1, \mu = 0)$ and $(\lambda = 0, \mu = 1)$. In the first case, one obtains

$$P(x, x+1, \dots, x+n, t+1) = (1-\alpha)\sum_{m=0}^{n}\alpha^m$$
$$\times P(\dots, x+m-2, x+m, \dots, x+n, t)$$
$$+ \alpha^{n+1} P(x-1, \dots, x+n-1, t). \quad (7)$$

It is seen that the rate of particles all hopping to right is simply the rate of one particle hopping to right, to the power of the the number of particles. This shows that there is no pushing. This is the simple exclusion process.

For the second case, one obtains

$$P(x, x+1, \dots, x+n, t+1) = (1-\alpha)^{n+1} P(x, \dots, x+n, t)$$
$$+ \alpha\sum_{m=0}^{n}(1-\alpha)^{n-m}$$
$$\times P(\dots, x+m-1, x+m+1, \dots, x+n, t). \quad (8)$$

This shows that there is a pushing process, the probability of which does not depend on the length of the block:

$$A\cdots A\emptyset \to \emptyset A\cdots A, \qquad \text{with the probability } \alpha. \quad (9)$$

This is the drop-push model.

3 The conditional probability

The n-particle analogue of (2), can be written as

$$P(\mathbf{x}, t+1) = (UP)(\mathbf{x}, t),$$
$$= \left[(1-\alpha+\alpha T_1)\cdots(1-\alpha+\alpha T_n)P\right](\mathbf{x}, t), \quad (10)$$

where

$$(T_j P)(x_1, \dots, x_n, t) :=$$
$$P(x_1, \dots, x_j - 1, \dots, x_n, t). \quad (11)$$

For the evolution equation (10), the Bethe-ansatz solution (the eigenvector of U) corresponding to the eigenvalue u is

$$u\Psi(\mathbf{x}) = \left[(1-\alpha+\alpha T_1)\cdots(1-\alpha+\alpha T_n)\Psi\right](\mathbf{x}), \quad (12)$$

subject to the condition

$$\Psi(\dots, x_j = x, x_{j+1} = x, \dots) =$$
$$\lambda\Psi(\dots, x_j = x, x_{j+1} = x+1, \dots)$$
$$+ \mu\Psi(\dots, x_j = x-1, x_{j+1} = x, \dots). \quad (13)$$

Using the Bethe-ansatz

$$\Psi_{\mathbf{k}}(\mathbf{x}) = \sum_{\sigma} A_{\sigma} e^{i\,\mathbf{x}\cdot\sigma(\mathbf{k})}, \tag{14}$$

where σ runs over n-permutations and

$$A_1 = 1, \tag{15}$$

one arrives at

$$u = \prod_{j=1}^{n} \left(1 - \alpha + \alpha\, e^{-i\,k_j}\right), \tag{16}$$

and

$$A_{\sigma\,\sigma_j} = S\left(k_{\sigma(j)}, k_{\sigma(j+1)}\right) A_{\sigma}, \tag{17}$$

where σ_j changes j to $j+1$ and $j+1$ to j, and leaves the other numbers between 1 and n intact, and

$$S\left(k_1, k_2\right) = -\frac{1 - \lambda\, e^{i\,k_2} - \mu\, e^{-i\,k_1}}{1 - \lambda\, e^{i\,k_1} - \mu\, e^{-i\,k_2}}. \tag{18}$$

This derivation is essentially the same as that used in [1–3]. Using these, the conditional probability (of finding the particles at $\mathbf{x}$ at the t, when they have been at $\mathbf{y}$ at the 0) is obtained as

$$P(\mathbf{x}, t; \mathbf{y}, 0) = \int \frac{\mathrm{d}^n k}{(2\pi)^n}\, \Psi_{\mathbf{k}}(\mathbf{x})\, e^{-i\,\mathbf{k}\cdot\mathbf{y}}\, u^t, \tag{19}$$

where the integration runs from 0 to 2π, for each of the k_j's. Also, to treat the singularity arising from S in Ψ, one is supposed to multiply λ and μ in the denominator by $e^{-\epsilon}$. It is seen that the right-hand side is equal $\delta_{\mathbf{x},\mathbf{y}}$ (for $\mathbf{x}$ and $\mathbf{y}$ in the physical region). So, the right-hand side satisfies the appropriate initial condition and evolution equation for the conditional probability, and hence is the (unique) solution to the conditional probability.

4 The drift rates

In the two-particle sector, the one-particle probabilities are defined as

$$P_1(x, t) := \sum_{x_2 > x} P(x, x_2, t)$$
$$P_2(x, t) := \sum_{x_1 < x} P(x_1, x, t). \tag{20}$$

From these,

$$
\begin{aligned}
P_1(x, t+1) &= (1-\alpha)\, P_1(x, t) + \alpha\, P_1(x-1, t)\\
&\quad + \lambda\,\alpha\,(1-\alpha)\,[P(x, x+1, t) - P(x-1, x, t)],\\
P_2(x, t+1) &= (1-\alpha)\, P_2(x, t) + \alpha\, P_2(x-1, t)\\
&\quad + \mu\,\alpha\,(1-\alpha)\,[P(x-2, x-1, t) - P(x-1, x, t)]. \tag{21}
\end{aligned}
$$

Defining

$$\langle X_1\rangle(t) := \sum_x x\, P_1(x, t),$$
$$\langle X_2\rangle(t) := \sum_x x\, P_2(x, t), \tag{22}$$

(the expectation value of the position of the first and second particles) one has

$$
\begin{aligned}
\langle X_1\rangle(t+1) &= \langle X_1\rangle(t) + \alpha - \lambda\,\alpha\,(1-\alpha)\, P_r(1, t),\\
\langle X_2\rangle(t+1) &= \langle X_2\rangle(t) + \alpha + \mu\,\alpha\,(1-\alpha)\, P_r(1, t), \tag{23}
\end{aligned}
$$

where

$$P_r(x, t) := \sum_y P(y, y+x, t). \tag{24}$$

Writing (19) for the two-particle sector,

$$
\begin{aligned}
P(x_1, x_2, t; y_1, y_2, 0) &= \int \frac{\mathrm{d}^2 k}{(2\pi)^2}\, \Big[e^{i\,(k_1\, x_1 + k_2\, x_2)}\\
&\quad - \frac{1 - \lambda\, e^{i\,k_2} - \mu\, e^{-i\,k_1}}{1 - \lambda\, e^{i\,k_1} - \mu\, e^{-i\,k_2}}\, e^{i\,(k_2\, x_1 + k_1\, x_2)} \Big]\, e^{-i\,(k_1\, y_1 + k_2\, y_2)}\\
&\quad \times \left(1 - \alpha + \alpha\, e^{-i\,k_1}\right)^t \left(1 - \alpha + \alpha\, e^{-i\,k_2}\right)^t, \tag{25}
\end{aligned}
$$

one arrives at

$$
\begin{aligned}
P_r(x, t) &= \int \frac{\mathrm{d}k}{2\pi}\, \left[e^{i\,k\,x} + e^{-i\,k\,(x-1)} \right] e^{-i\,k\,(y_2 - y_1)}\\
&\quad \times \left(1 - \alpha + \alpha\, e^{-i\,k}\right)^t \left(1 - \alpha + \alpha\, e^{i\,k}\right)^t, \tag{26}
\end{aligned}
$$

where, using (24), the summation over y is done, which leads to a delta function $\delta(k_1 + k_2)$, using which one of the integrations is carried out.

A steepest descent calculation shows that if t is large and x is not large,

$$P_r(x, t) \sim \frac{1}{\sqrt{\pi\,\alpha\,(1-\alpha)\,t}}. \tag{27}$$

So, for large t,

$$
\begin{aligned}
\langle X_1\rangle(t) &= \langle X_1\rangle(0) + \alpha\, t - \lambda\, \left[2\sqrt{\frac{\alpha\,(1-\alpha)\,t}{\pi}} + C + o(1) \right],\\
\langle X_2\rangle(t) &= \langle X_2\rangle(0) + \alpha\, t + \mu\, \left[2\sqrt{\frac{\alpha\,(1-\alpha)\,t}{\pi}} + C + o(1) \right]. \tag{28}
\end{aligned}
$$

One also has

$$
\begin{aligned}
(\langle X_2\rangle - \langle X_1\rangle)(t+1) &=\\
(\langle X_2\rangle - \langle X_1\rangle)(t) &+ \alpha\,(1-\alpha)\, P_r(1, t),\\
\langle X\rangle(t+1) &= \langle X\rangle(t) + \alpha\\
&\quad + \frac{\mu - \lambda}{2}\,\alpha\,(1-\alpha)\, P_r(1, t),\\
(\mu\,\langle X_1\rangle + \lambda\,\langle X_2\rangle)(t+1) &= (\mu\,\langle X_1\rangle + \lambda\,\langle X_2\rangle)(t) + \alpha, \tag{29}
\end{aligned}
$$

where

$$\langle X \rangle := \frac{1}{2} \left(\langle X_1 \rangle + \langle X_2 \rangle \right), \tag{30}$$

is the expectation value of the position of the particles. So for all times

$$(\mu \langle X_1 \rangle + \lambda \langle X_2 \rangle)(t) = (\mu \langle X_1 \rangle + \lambda \langle X_2 \rangle)(0) + \alpha t, \tag{31}$$

and for large times,

$$\begin{aligned}
(\langle X_2 \rangle - \langle X_1 \rangle)(t) &= (\langle X_2 \rangle - \langle X_1 \rangle)(0) \\
&\quad + 2\sqrt{\frac{\alpha(1-\alpha)t}{\pi}} + C + o(1), \\
\langle X \rangle(t) &= \langle X \rangle(0) + \alpha t \\
&\quad + (\mu - \lambda) \left[\sqrt{\frac{\alpha(1-\alpha)t}{\pi}} + C + o(1) \right],
\end{aligned}$$
$$\tag{32}$$

where C is a constant. So the drift rates at large t are

$$\begin{aligned}
V_1 &:= \frac{\mathrm{d}\langle X_1 \rangle}{\mathrm{d}t} = \alpha - \lambda \sqrt{\frac{\alpha(1-\alpha)}{\pi t}}, \\
V_2 &:= \frac{\mathrm{d}\langle X_2 \rangle}{\mathrm{d}t} = \alpha + \mu \sqrt{\frac{\alpha(1-\alpha)}{\pi t}}.
\end{aligned}
\tag{33}$$

$\langle X_1 \rangle$ and $\langle X_2 \rangle$ are the expectation values of the positions of the first and second particles, respectively, and V_1 and V_2 are their corresponding velocities. As t is discrete, these velocities are defined only when the X_i's are smooth functions of t, which happens at large times.

The above equations show that the drift velocities of both particles approach α for large times. The reason is that at large times the particles are far from each other and effectively do not interact with each other. But the next leading terms in velocities are negative for the first particle and positive for the second particle, which is expected from the hindering effect of the second particle on the first, and the pushing effect of the first particle on the second. One can see that the results obtained in [3] are recovered, provided one replaces $\alpha(1-\alpha)t$ with t.

5 The diffusion rate

Starting from (21), and defining

$$\begin{aligned}
\langle X_1^2 \rangle(t) &:= \sum_x x^2 P_1(x,t), \\
\langle X_2^2 \rangle(t) &:= \sum_x x^2 P_2(x,t),
\end{aligned}
\tag{34}$$

one has

$$\begin{aligned}
\langle X_1^2 \rangle(t+1) &= \langle X_1^2 \rangle(t) + 2\alpha \langle X_1 \rangle(t) + \alpha \\
&\quad - \lambda \alpha (1-\alpha) \sum_x (2x+1) P(x,x+1,t), \\
\langle X_2^2 \rangle(t+1) &= \langle X_2^2 \rangle(t) + 2\alpha \langle X_2 \rangle(t) + \alpha \\
&\quad + \mu \alpha (1-\alpha) \sum_x (2x+3) P(x,x+1,t).
\end{aligned}
\tag{35}$$

Defining

$$\begin{aligned}
\langle X^2 \rangle &:= \frac{1}{2} \left(\langle X_1^2 \rangle + \langle X_2^2 \rangle \right), \\
\Delta^2 &:= \langle X^2 \rangle - (\langle X \rangle)^2,
\end{aligned}
\tag{36}$$

(Δ^2 is the variance of the position of the particles) one arrives at

$$\begin{aligned}
\Delta^2(t+1) &= \Delta^2(t) + \alpha(1-\alpha) + \alpha(1-\alpha)(\mu-\lambda) \\
&\quad \times \left[\frac{1}{2} \sum_x (2x+1) P(x,x+1,t) - P_r(1,t)\langle X \rangle(t) \right] \\
&\quad + \alpha(1-\alpha)[\mu - \alpha(\mu-\lambda)] P_r(1,t) \\
&\quad - \frac{\alpha^2(1-\alpha)^2(\mu-\lambda)^2}{4} P_r^2(1,t).
\end{aligned}
\tag{37}$$

From (27), it is seen that the last two terms on the right-hand side vanish as $t \to \infty$. One also has

$$\begin{aligned}
\sum_x (2x+1) P(x,x+1,t) &= \sum_x \int \frac{\mathrm{d}^2 k}{4\pi^2} (2x+1) e^{i(k_1+k_2)x} \\
&\quad \times \left[e^{ik_2} - \frac{1 - \lambda e^{ik_2} - \mu e^{-ik_1}}{1 - \lambda e^{ik_1} - \mu e^{-ik_2}} e^{ik_1} \right] e^{-i(k_1 y_1 + k_2 y_2)} \\
&\quad \times \left(1 - \alpha + \alpha e^{-ik_1} \right)^t \left(1 - \alpha + \alpha e^{-ik_2} \right)^t, \\
&= \int \frac{\mathrm{d}^2 k}{4\pi^2} \left[2\pi \delta(k_1+k_2) - 4\pi i \delta'(k_1+k_2) \right] \\
&\quad \times \left[e^{ik_2} - \frac{1 - \lambda e^{ik_2} - \mu e^{-ik_1}}{1 - \lambda e^{ik_1} - \mu e^{-ik_2}} e^{ik_1} \right] e^{-i(k_1 y_1 + k_2 y_2)} \\
&\quad \times \left(1 - \alpha + \alpha e^{-ik_1} \right)^t \left(1 - \alpha + \alpha e^{-ik_2} \right)^t.
\end{aligned}
\tag{38}$$

Here δ' is the derivative of δ with respect to its argument. We are seeking those terms on the right-hand side of (37), which don't vanish as $t \to \infty$. On the right-hand side of (38), it is seen that the terms proportional to $\delta(k_1+k_2)$ in the integrand, give rise to terms proportional to $t^{-1/2}$ (for large t). (In fact these terms are proportional to $P_r(1,t)$.) Denoting the terms coming from that part of the integrand which is proportional to $\delta'(k_1+k_2)$ by I, one has

$$\begin{aligned}
I &= \int \frac{i \mathrm{d}^2 k}{\pi} \delta(k_1+k_2) \frac{\partial}{\partial k_2} \\
&\quad \times \left\{ \left[e^{ik_2} - \frac{1 - \lambda e^{ik_2} - \mu e^{-ik_1}}{1 - \lambda e^{ik_1} - \mu e^{-ik_2}} e^{ik_1} \right] \times e^{-i(k_1 y_1 + k_2 y_2)} \right. \\
&\quad \left. \times \left(1 - \alpha + \alpha e^{-ik_1} \right)^t \left(1 - \alpha + \alpha e^{-ik_2} \right)^t \right\}.
\end{aligned}
\tag{39}$$

Differentiation of the fraction, $S(k_1, k_2)$, needs some care. The derivative of this fraction is singular at $k_1 = k_2 = 0$. To remove this singularity, one has to replace k_1 with $k_1 + i\epsilon$ in the denominator. This prescription guaranties that in (25), the integral of the second term on the right-hand side tends to zero as x_2 tends to infinity. We are

interested in the behavior of I for large times, and this is determined by the behavior of the integrand multiplier of $\delta(k_1 + k_2)$ for $k_1 = -k_2$, where k_2 is small. One has

$$\frac{\partial}{\partial k_2} \left(\frac{1 - \lambda e^{i k_2} - \mu e^{i k_1}}{1 - \lambda e^{i k_1} - \mu e^{i k_2}} \right) \Bigg|_{k_1 = -k_2 = -k \sim 0} =$$

$$- i \left. \frac{\lambda e^{i k} - \mu}{1 - e^{i k}} \right|_{k \sim 0}, \quad = \left. \frac{\mu - \lambda}{k - i \epsilon} \right|_{k \sim 0},$$

$$= (\mu - \lambda) \operatorname{pf} \left(\frac{1}{k} \right) + i \pi (\mu - \lambda) \delta(k). \quad (40)$$

Here pf means a pseudo-function (the Cauchy principal value in integration). Putting this in (39), and keeping only terms which don't vanish as $t \to \infty$, one arrives at

$$I = \int \frac{i \, \mathrm{d}k}{\pi} \left[-i \pi (\mu - \lambda) \delta(k) + 2 (-i \alpha t) \right.$$

$$\times \left. \left(1 - \alpha + \alpha e^{i k} \right)^t \left(1 - \alpha + \alpha e^{-i k} \right)^t \right] + \cdots,$$

$$= (\mu - \lambda) + \frac{2 \alpha t}{\sqrt{\pi \alpha (1 - \alpha) t}} + \cdots \quad (41)$$

So,

$$\sum_x (2x + 1) P(x, x + 1, t) =$$

$$(\mu - \lambda) + \frac{2 \alpha t}{\sqrt{\pi \alpha (1 - \alpha) t}} + \cdots \quad (42)$$

Using (27) and (32), one arrives at

$$P_r(1, t) \langle X \rangle(t) = \frac{\mu - \lambda}{\pi} + \frac{\alpha t}{\sqrt{\pi \alpha (1 - \alpha) t}} + \cdots \quad (43)$$

Putting (42) and (43) in (37), one arrives at

$$\Delta^2(t + 1) =$$

$$\Delta^2(t) + \alpha (1 - \alpha) \left[1 + (\mu - \lambda)^2 \left(\frac{1}{2} - \frac{1}{\pi} \right) \right] + \cdots, \quad (44)$$

from which,

$$\lim_{t \to \infty} \frac{\mathrm{d} \Delta^2}{\mathrm{d} t} = \alpha (1 - \alpha) \left[1 + (\mu - \lambda)^2 \left(\frac{1}{2} - \frac{1}{\pi} \right) \right]. \quad (45)$$

This diffusion rate, is again defined only at large times, when one can treat Δ^2 as a function of continuous time. It is seen that it is in agreement with that obtained in [3], provided one replaces $\alpha (1 - \alpha) t$ with t.

6 Concluding remarks

The main result of the paper was to introduce a discrete-time discrete-space model, solvable through the Bethe-ansatz method. The model contains a free parameter (say λ) that for certain values reproduces the simple exclusion model and the drop-push model. The conditional probability of finding particles at different sites was obtained as a function of time, from which in principle one can derive any correlation function. The conditional probabilities were calculated for the general multi-particle sector. The drift- and diffusion-rates, however, were explicitly calculated only for the two-particle sector, and it was shown that the results agreed with those of the continuous-time system at large-times. There remains the question of performing similar calculations for the multi-particle sector. For large times, one can put forward the following arguments. For large times, only the behavior of the integrand in (19) around $\mathbf{k} = \mathbf{0}$ is important, and it is seen that for $\lambda = \mu$, there is no pole in the scattering matrix S. In fact, S becomes one for $\lambda = \mu$ and $\mathbf{k} = \mathbf{0}$, which shows that for $\lambda = \mu$ and for large times, the conditional probability takes the form

$$P(\mathbf{x}, t; \mathbf{y}, 0) = \sum_\sigma P_0[\sigma(\mathbf{x}), t; \mathbf{y}, 0], \quad (46)$$

where P_0 is the conditional probability of a system consisting of free particles hopping to the right, which corresponds to the one obtained with $S = 0$ in (19). This shows that for $\lambda = \mu$, at large times the system behaves collectively as a collection of free particles. Hence the drift- and diffusion-rates should be α and $\alpha(1 - \alpha)$ respectively, which agrees with the particular case of the two-particles.

For the general case $\lambda \neq \mu$, at large times the particles will generally be far from each other. The interaction terms coming from the scattering matrix, are in the from of products of two-particle scattering matrices. So it is plausible that for large times and for calculating up to 2-point functions, one neglects more-than two-particle interactions and interactions between non-adjacent particles. Then, from the $n!$ terms in the Bethe-ansatz solution for the n-particle sector, there remains only n terms. The drift rate at large times is expected to remain α again. For the diffusion rate, one can argue that it should be $\alpha(1 - \alpha)$ (the free-particle value) plus a function of $(\mu - \lambda)$ which vanishes at $\mu = \lambda$. The additional term comes from the interaction of the neighboring particles. So it is expected to be proportional to $(\lambda - \mu)^2$. This shows that the drift- and diffusion-rates obtained for the two-particle sector at large times, serve as qualitative results for the multi-particle sector as well.

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